

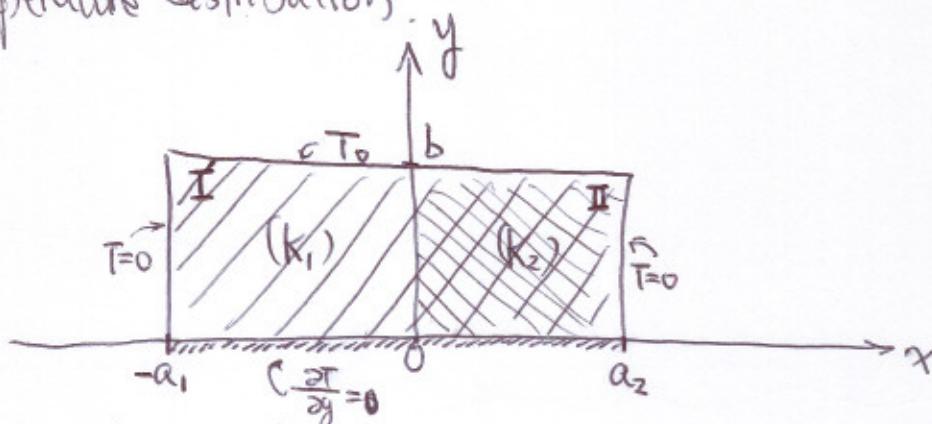
2.5. Advanced Steady-State Conduction Problems.

2.5.1. Composite Problem

Consider a composite material shown in figure.

The thermal conductivity of the left-side material is k_1 , and the thermal conductivity of the right-side material is k_2 .

The edges of the material parallel to y -axis are maintained at $T=0$; the lower edge is insulated, and the upper edge is maintained at $T=T_0$. Find the steady-state temperature distribution.



The complete problem:

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0$$

B.C. $\left. T \right|_{x=-a_1} = 0$

$\left. T \right|_{x=a_2} = 0$

$\left. \frac{\partial T}{\partial y} \right|_{y=0} = 0$

$T \Big|_{y=b} = T_0$ ← Nonhomogeneous

Interface condition:

$$\begin{cases} \left. T_I \right|_{x=0} = \left. T_{II} \right|_{x=0} \\ \left. +k_1 \frac{\partial T_I}{\partial x} \right|_{x=0} = -\left. k_2 \frac{\partial T_{II}}{\partial x} \right|_{x=0} \end{cases}$$

i.e. $\left. T \right|_{x=0^-} = \left. T \right|_{x=0^+}$

$$\left. -k_1 \frac{\partial T}{\partial x} \right|_{x=0^-} = -\left. k_2 \frac{\partial T}{\partial x} \right|_{x=0^+}$$

① Separation of $T(x, y)$.

Assuming a product solution: $T(x, y) = X(x)Y(y)$

$$\text{so: } \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \mu \quad (\text{const.})$$

$$\begin{aligned} \text{i.e.: } & \left\{ \begin{array}{l} X''(x) - \mu X(x) = 0 \\ Y''(y) + \mu Y(y) = 0 \end{array} \right. \end{aligned}$$

Because of the different thermal properties of the two sections of the composite material (separated at $x=0$), we need to solve the equation for $X(x)$ independently over two intervals, $-a_1 < x < 0$ and $0 < x < a_2$, and then fit these two solutions together so that both the temperature and the heat flux are continuous at $x=0$.

② Solving ODEs.

There are three cases for μ :

$$(1) \underline{\text{For } \mu > 0} \quad \text{let. } \mu = \lambda^2$$

$$\text{so: } \left\{ \begin{array}{l} X''(x) - \lambda^2 X(x) = 0 \\ Y''(y) + \lambda^2 Y(y) = 0 \end{array} \right.$$

solution for $X(x)$:

* There are two solutions:

$$\begin{cases} X_I(x) = A_1 \cosh \lambda x + B_1 \sinh \lambda x & (-a < x < 0) \end{cases}$$

$$\begin{cases} X_{II}(x) = A_2 \cosh \lambda x + B_2 \sinh \lambda x & (0 < x < a_2) \end{cases}$$

Imposing B.C.: $T|_{x=-a_1} = 0 \Rightarrow X_I|_{x=-a_1} = 0$

$$\text{so: } A_1 \cosh \lambda a_1 + B_1 \sinh \lambda (-a_1) = 0$$

$$\text{i.e.: } A_1 \cosh \lambda a_1 = B_1 \sinh \lambda a_1$$

$$\text{and: } X_I(x) = A_1 \cosh \lambda x + A_1 \cdot \frac{\cosh \lambda a_1}{\sinh \lambda a_1} \cdot \sinh \lambda x$$

$$\text{define: } C_1 = \frac{A_1}{\sinh \lambda a_1} \text{ (remains arbitrary constant)}$$

$$\text{so: } X_I(x) = C_1 \sinh \lambda a_1 \cosh \lambda x + C_1 \cosh \lambda a_1 \sinh \lambda x$$

therefore, $X_I(x) = C_1 \sinh \lambda (a_1 + x)$

Imposing B.C.: $T|_{x=a_2} = 0 \Rightarrow X_{II}|_{x=a_2} = 0$

$$\text{so: } A_2 \cosh \lambda a_2 + B_2 \sinh \lambda a_2 = 0$$

$$\text{i.e.: } A_2 \cosh \lambda a_2 = -B_2 \sinh \lambda a_2$$

$$\text{and: } X_{II}(x) = A_2 \cosh \lambda x - A_2 \cdot \frac{\cosh \lambda a_2}{\sinh \lambda a_2} \sinh \lambda x$$

$$\text{define: } C_2 = \frac{A_2}{\sinh \lambda a_2} \text{ (remains arbitrary constant)}$$

$$\text{so: } X_{II}(x) = C_2 \sinh \lambda a_2 \cosh \lambda x - C_2 \cosh \lambda a_2 \sinh \lambda x$$

therefore, $X_{II}(x) = C_2 \sinh \lambda (a_2 - x)$

Impose Interface Condition: $T_I|_{x=0} = T_{II}|_{x=0}$

$$\text{i.e.: } X_I \Big|_{x=0} \stackrel{Y(x)}{=} X_{II} \Big|_{x=0} \stackrel{Y(y)}{=}$$

$$\text{so: } C_1 \sinh \lambda a_1 = C_2 \sinh \lambda a_2$$

$$\text{or: } \frac{C_1}{\sinh \lambda a_2} = \frac{C_2}{\sinh \lambda a_1} \equiv K \quad (\text{arbitrary constant})$$

$$\text{therefore: } \left\{ \begin{array}{l} X_I(x) = K \sinh \lambda a_2 \sinh \lambda (a_1 + x) \\ X_{II}(x) = K \sinh \lambda a_1 \sinh \lambda (a_2 - x) \end{array} \right.$$

Impose Interface Condition:

$$-k_1 \frac{\partial T_I}{\partial x} \Big|_{x=0} = -k_2 \frac{\partial T_{II}}{\partial x} \Big|_{x=0}$$

$$\text{i.e.: } -k_1 \frac{dX_I}{dx} \Big|_{x=0} = -k_2 \frac{dX_{II}}{dx} \Big|_{x=0}$$

$$\text{so: } k_1 \lambda K \sinh \lambda a_2 \cosh \lambda a_1 = -k_2 \lambda K \sinh \lambda a_1 \cosh \lambda a_2$$

$$\text{or: } \lambda K (k_1 \sinh \lambda a_2 \cosh \lambda a_1 + k_2 \sinh \lambda a_1 \cosh \lambda a_2) = 0$$

$$\text{Therefore: } \boxed{K = 0} > 0$$

$$\text{and } \left\{ \begin{array}{l} X_I(x) = 0 \\ X_{II}(x) = 0 \end{array} \right.$$

not a meaningful solution

Conclusion: λ cannot be greater than 0!

(2) For $\mu = 0$

$$\text{So: } \begin{cases} X''(x) = 0 \\ Y''(y) = 0 \end{cases}$$

Solution for $X(x)$:

* There are two solutions.

$$\begin{cases} X_I(x) = A_1 x + B_1 & (-a_1 < x < 0) \\ X_{II}(x) = A_2 x + B_2 & (0 < x < a_2) \end{cases}$$

$$\text{Imposing B.C. } T|_{x=-a_1} = 0 \Rightarrow X_I|_{x=-a_1} = 0$$

$$\text{So. } A_1(-a_1) + B_1 = 0 \Rightarrow B_1 = A_1 a_1$$

$$\text{i.e.: } \underline{X_I(x) = A_1(a_1+x)}$$

$$\text{Imposing B.C. } T|_{x=a_2} = 0 \Rightarrow X_{II}|_{x=a_2} = 0$$

$$\text{So. } A_2 a_2 + B_2 = 0 \Rightarrow B_2 = -A_2 a_2$$

$$\text{i.e.: } \underline{X_{II}(x) = A_2(x-a_2)}$$

$$\text{Imposing Interface Condition: } T_I|_{x=0} = T_{II}|_{x=0}$$

$$\text{i.e.: } \underline{X_I|_{x=0} = X_{II}|_{x=0}}$$

$$\text{So: } A_1 a_1 = -A_2 a_2$$

$$\text{i.e.: } \frac{A_1}{a_2} = -\frac{A_2}{a_1} \equiv K \quad (\text{arbitrary constant})$$

$$\text{therefore: } \begin{cases} X_I(x) = K a_2 (a_1 + x) \\ X_{II}(x) = K a_1 (a_2 - x) \end{cases}$$

Imposing Interface Condition. $-k_1 \frac{\partial T_I}{\partial x} \Big|_{x=0} = -k_2 \frac{\partial T_{II}}{\partial x} \Big|_{x=0}$

i.e.: $-k_1 \frac{dX_I}{dx} \Big|_{x=0} = -k_2 \frac{dX_{II}}{dx} \Big|_{x=0}$

So: $k_1 K Q_2 = -k_2 K Q_1$

or: $K \underbrace{(k_1 Q_2 + k_2 Q_1)}_{>0} = 0$

Therefore: $K = 0$

and: $\begin{cases} X_I(x) = 0 \\ X_{II}(x) = 0 \end{cases}$

not a meaningful solution.

Conclusion: $\mu \neq 0$!

(3) For $\mu < 0$ Let $\mu = -\lambda^2$ ($\lambda > 0$)

so: $\begin{cases} X''(x) + \lambda^2 X(x) = 0 \\ Y''(y) - \lambda^2 Y(y) = 0 \end{cases}$

Solution for $X(x)$:

* There are two solutions,

$$\begin{cases} X_I(x) = A_1 \cos \lambda x + B_1 \sin \lambda x & (-a < x < 0) \\ X_{II}(x) = A_2 \cosh \lambda x + B_2 \sinh \lambda x & (0 < x < a) \end{cases}$$

Solution for $Y(y)$:

$$Y(y) = D \cosh \lambda y + E \sinh \lambda y$$

Impose B.C.: $\frac{\partial Y}{\partial y}|_{y=0} = 0 \Rightarrow \frac{dY}{dy}|_{y=0} = 0$

$$\frac{dY}{dy}|_{y=0} = \lambda D \sinh(0) + \lambda E \cosh(0) = 0$$

$$\text{i.e.: } E = 0$$

And: $\boxed{Y(y) = D \cosh \lambda y}$

Impose B.C.: $T|_{x=-a_1} = 0 \Rightarrow X_I|_{x=-a_1} = 0$

$$\text{So: } A_1 \cos \lambda (-a_1) + B_1 \sin \lambda (-a_1) = 0$$

$$\text{i.e.: } A_1 \cos \lambda a_1 = B_1 \sin \lambda a_1$$

$$\text{or: } \frac{A_1}{\sin \lambda a_1} = \frac{B_1}{\cos \lambda a_1} \equiv C_1 \quad (\text{arbitrary constant})$$

$$\text{So: } X_I(x) = C_1 \sin \lambda a_1 \cos \lambda x + C_1 \cos \lambda a_1 \sin \lambda x$$

therefore: $\boxed{X_I(x) = C_1 \sin \lambda (a_1 + x)}$

Impose B.C.: $T|_{x=a_2} = 0 \Rightarrow X_{II}|_{x=a_2} = 0$

$$\text{So: } A_2 \cos \lambda a_2 + B_2 \sin \lambda a_2 = 0$$

$$\text{i.e.: } \frac{A_2}{\sin \lambda a_2} = \frac{-B_2}{\cos \lambda a_2} = C_2 \quad (\text{arbitrary constant})$$

$$\text{So: } X_{II}(x) = C_2 \sin \lambda a_2 \cos \lambda x - C_2 \cos \lambda a_2 \sin \lambda x$$

therefore: $\boxed{X_{II}(x) = C_2 \sin \lambda (a_2 - x)}$

Imposing Interface Condition: $T_I|_{x=0} = T_{II}|_{x=0}$

$$\text{i.e. } X_I|_{x=0} = X_{II}|_{x=0}$$

$$\text{so: } C_1 \sin \lambda a_1 = C_2 \sin \lambda a_2$$

$$\text{or. } \frac{C_1}{\sin \lambda a_2} = \frac{C_2}{\sin \lambda a_1} = K \quad (\text{arbitrary constant})$$

therefore,
$$\begin{cases} X_I(x) = K \sin \lambda a_2 \sin \lambda(a_1 + x) \\ X_{II}(x) = K \sin \lambda a_1 \sin \lambda(a_2 - x) \end{cases}$$

Imposing Interface Condition: $-k_1 \frac{\partial T_I}{\partial x}|_{x=0} = -k_2 \frac{\partial T_{II}}{\partial x}|_{x=0}$

$$\text{i.e. } -k_1 \frac{dX_I}{dx}|_{x=0} = -k_2 \frac{dX_{II}}{dx}|_{x=0}$$

$$\text{so: } k_1 K \sin \lambda a_2 \cos \lambda a_1 = -k_2 \lambda K \sin \lambda a_1 \cos \lambda a_2$$

$$\text{or: } \lambda K (k_1 \sin \lambda a_2 \cos \lambda a_1 + k_2 \sin \lambda a_1 \cos \lambda a_2) = 0$$

There are two possibilities.

If $K = 0 \Rightarrow \begin{cases} X_I(x) = 0 \\ X_{II}(x) = 0 \end{cases} \Rightarrow$ not a meaningful solution

We must have:

$$k_1 \sin \lambda a_2 \cos \lambda a_1 + k_2 \sin \lambda a_1 \cos \lambda a_2 = 0$$

This is the characteristic equation that determines eigenvalues: λ_n

③ Making final solution.

We have. $\begin{cases} X_I(x) = K \sin \lambda a_2 \sin \lambda (a_1 + x) & (-a_1 < x < 0) \\ X_{II}(x) = K \sin \lambda a_1 \sin \lambda (a_2 - x) & (0 < x < a_2) \end{cases}$

and $Y(y) = D \cosh \lambda y$

Therefore, for each $\lambda = \lambda_n \quad n=1, 2, 3, \dots$

$$T_n(x, y) = \begin{cases} D_n \sin \lambda_n a_2 \sin \lambda_n (a_1 + x) \cosh \lambda_n y & (-a_1 < x < 0) \\ D_n \sin \lambda_n a_1 \sin \lambda_n (a_2 - x) \cosh \lambda_n y & (0 < x < a_2) \end{cases}$$

And: $T(x, y) = \sum_{n=1}^{\infty} T_n(x, y)$

With λ_n defined by: $K_1 \sin \lambda a_2 \cos \lambda a_1 + K_2 \sin \lambda a_1 \cos \lambda a_2 = 0$.

④ Determining the unknown coefficient.

Applying nonhomogeneous B.C. $T|_{y=b} = T_0$

so: $T_0 = \sum_{n=1}^{\infty} (D_n \cosh \lambda_n b) \cdot X_n(x)$

here: $X_n(x) = \begin{cases} \sin \lambda_n a_2 \sin \lambda_n (a_1 + x) & (-a_1 < x < 0) \\ \sin \lambda_n a_1 \sin \lambda_n (a_2 - x) & (0 < x < a_2) \end{cases}$

(Note: All arbitrary constants are merged with D_n)

Note: Because of the interface conditions, $\bar{X}_n(x)$ is not orthogonal over entire interval $(-a_1, a_2)$. However, since the eigenvalue problem is satisfied on separate intervals $(-a_1, 0)$ and $(0, a_2)$, one can find a weight function to make $\bar{X}_n(x)$ orthogonal over entire interval.

i.e.: $\int_{-a_1}^{a_2} \bar{X}_n(x) \bar{X}_m(x) dx = 0 \quad \text{for } m \neq n$
does not hold!

but: $\int_{-a_1}^{a_2} p(x) \bar{X}_n(x) \bar{X}_m(x) dx = 0 \quad \text{for } m \neq n$
does hold!

where $p(x)$ is a weight function.

It can be proved (Sturm-Liouville problem) that for current problem:

$$p(x) = \begin{cases} k_1 & -a_1 < x < 0 \\ k_2 & 0 < x < a_2 \end{cases}$$

i.e., $\int_{-a_1}^{a_2} p(x) \bar{X}_n(x) \bar{X}_m(x) dx = 0 \quad \text{for } m \neq n$.

$$\text{So, } T_0 = \sum_{n=1}^{\infty} (D_n \cosh \lambda_n b) \cdot \bar{X}_n(x)$$

$$\int_{a_1}^{a_2} T_0 p(x) \bar{X}_m(x) dx = \sum_{n=1}^{\infty} \int_{a_1}^{a_2} (D_n \cosh \lambda_n b) \cdot \bar{X}_n(x) \bar{X}_m(x) p(x) dx$$

i.e., $T_0 \int_{-a_1}^{a_2} p(x) \bar{X}_n(x) dx = D_n \cosh \lambda_n b \int_{-a_1}^{a_2} p(x) \bar{X}_n^2(x) dx$

or:

$$D_n = \frac{T_0}{\cosh \lambda_n b} \cdot \frac{\int_{-a_1}^{a_2} p(x) \bar{X}_n(x) dx}{\int_{-a_1}^{a_2} p(x) \bar{X}_n^2(x) dx}$$

$$= \frac{T_0}{\cosh \lambda_n b} \cdot \frac{\int_{-a_1}^0 k_1 \sin \lambda_n a_2 \sin \lambda_n (a_1+x) dx + \int_0^{a_2} k_2 \sin \lambda_n a_1 \sin \lambda_n (a_2-x) dx}{\int_{-a_1}^0 k_1 \sin^3 \lambda_n a_2 \sin^2 \lambda_n (a_1+x) dx + \int_0^{a_2} k_2 \sin^3 \lambda_n a_1 \sin^2 \lambda_n (a_2-x) dx}$$

Therefore:

$$D_n = \frac{2T_0}{\lambda_n \cosh \lambda_n b} \cdot \frac{k_2 \sin \lambda_n a_1 + k_1 \sin \lambda_n a_2}{k_2 a_2 \sin^2 \lambda_n a_1 + k_1 a_1 \sin^2 \lambda_n a_2}$$

Note: for a medium consisting of M layers,

$$\frac{d^2 \bar{X}_i}{dx^2} + \lambda_i^2 \bar{X}_i = 0 \quad i=1, 2, \dots, M$$

for $x_i < x < x_{i+1}$ ↑
(label of layer)

The eigenfunction \bar{X}_i satisfy the following orthogonal relation.

$$\sum_{i=1}^M k_i \int_{x_i}^{x_{i+1}} \bar{X}_i(x) \bar{X}_{i'n}(x) dx = \begin{cases} 0 & n' \neq n \\ N_n & n' = n \end{cases}$$

(Weight function)